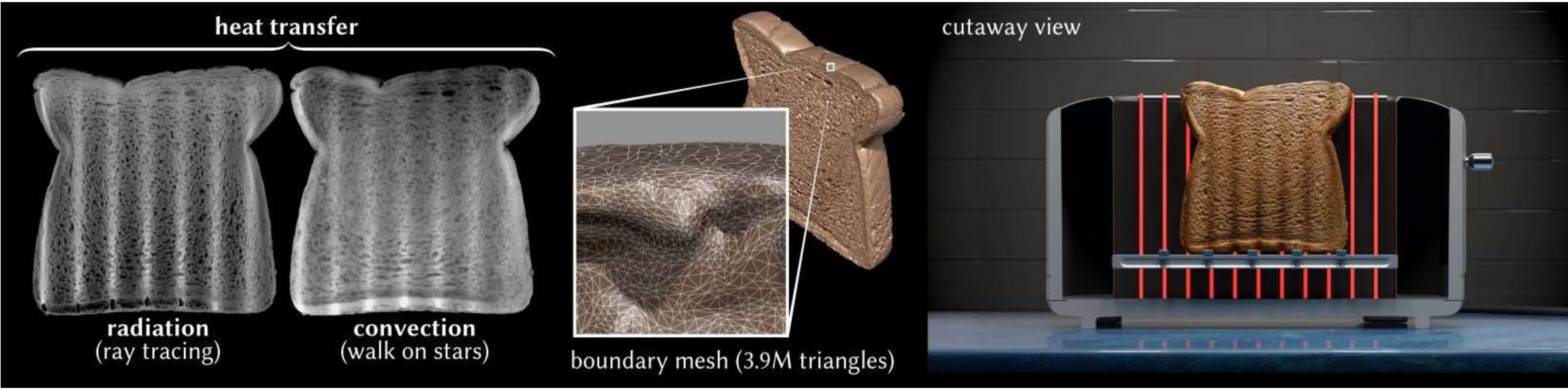


Walk on Stars

A Grid-Free Monte Carlo Method for PDEs with Neumann Boundary Conditions

Paper by: R. Sawhney, B. Miller, I. Gkioulekas, K, Crane
Presenter: **Siripon Sutthiwanna**

Overview

- *Review* | Walk on Sphere (WoS)
- *Contextualize* | Heat Diffusion problem.
- Boundary Conditions
- Walk on Star (WoSt)
- Applications
- Limitation

Delicious French toast photo obtained from
https://www.seriouseats.com/perfect-quick-easy-french-toast

Review | Walk on Sphere (WoS)

1. Initialize: $x^{(0)} = x$
2. While $\mathrm{distance}(x^{(n)} > \Gamma) > \varepsilon$
   a. Set $r_n = \mathrm{distance}(x^{(n)}, \Gamma)$
   b. Sample γ_n uniformly at the sphere centered in $x^{(n)}$ radius of r_n
   c. Set $x^{(n+1)} = x^{(n)} + r_n \gamma_n$
3. Else, when $\mathrm{distance}(x^{(n)} > \Gamma) \leq \varepsilon$
   a. $x_f =$ touching point at the boundary
   b. Return x_f as the estimator for x

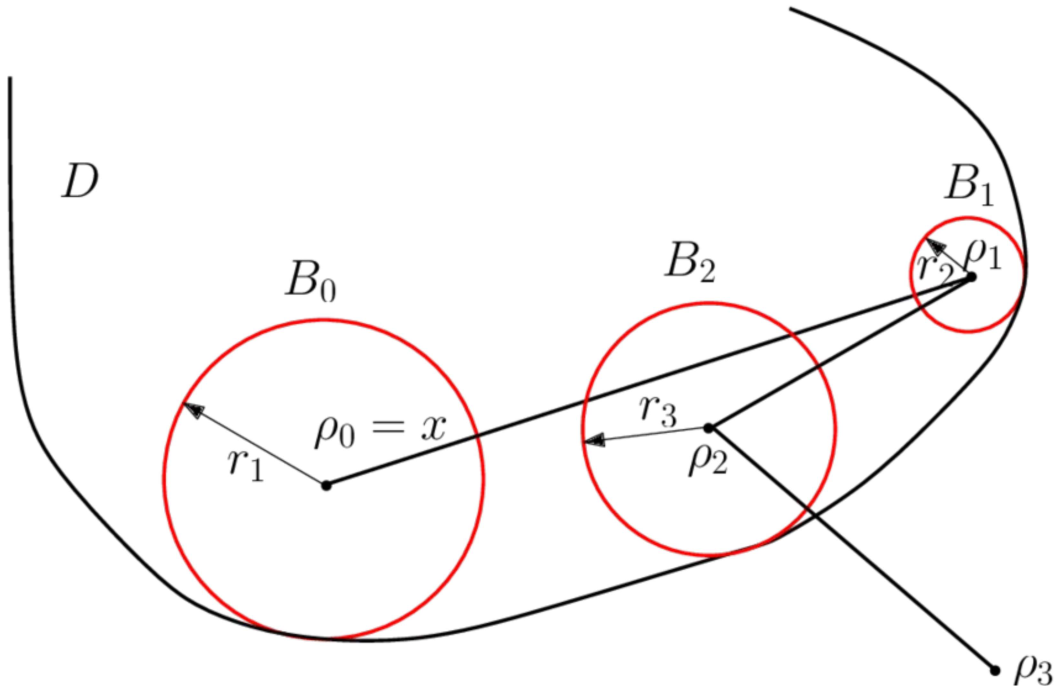


Image obtained from *AE Kyprianou, et al. - Unbiased `walk-on-spheres' Monte Carlo methods for the fractional Laplacian*

Review | Walk on Sphere (WoS) - *Why?*

- **no pre-computation**
- **agnostic to representation**
- parallelize-able
- easy to implement
- fast convergence
- easy to realize on most PDEs
- unbiased method
- robust to noise, numerically stable
- easy to compute div, grad, curl

method	linear FEM	Monte Carlo
#triangles	2M	**10M**
#samples	**47k nodes**	23k pixels
precompute	14 hours	**0.4 seconds**
solve	**13 seconds**	57 seconds

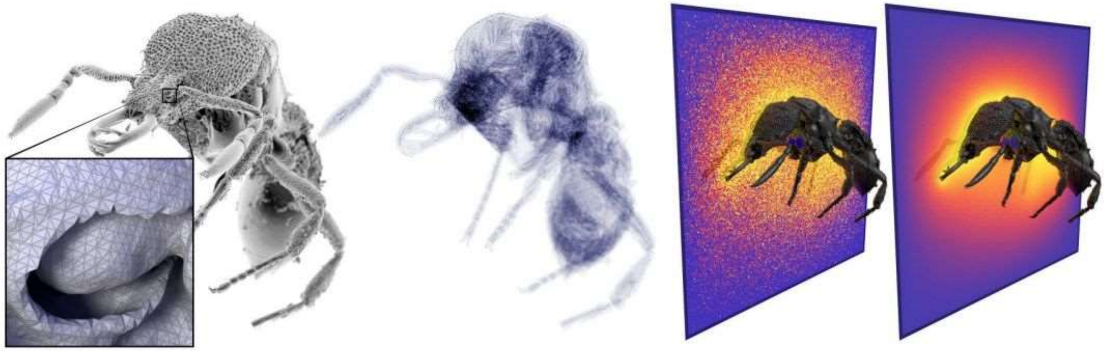


Image obtained from *R. Sawhney, K. Crane*
Monte Carlo Geometry Processing:
A Grid-Free Approach to PDE-based Methods on Volumetric Domains

Contextualize | Heat Diffusion Problem (1)

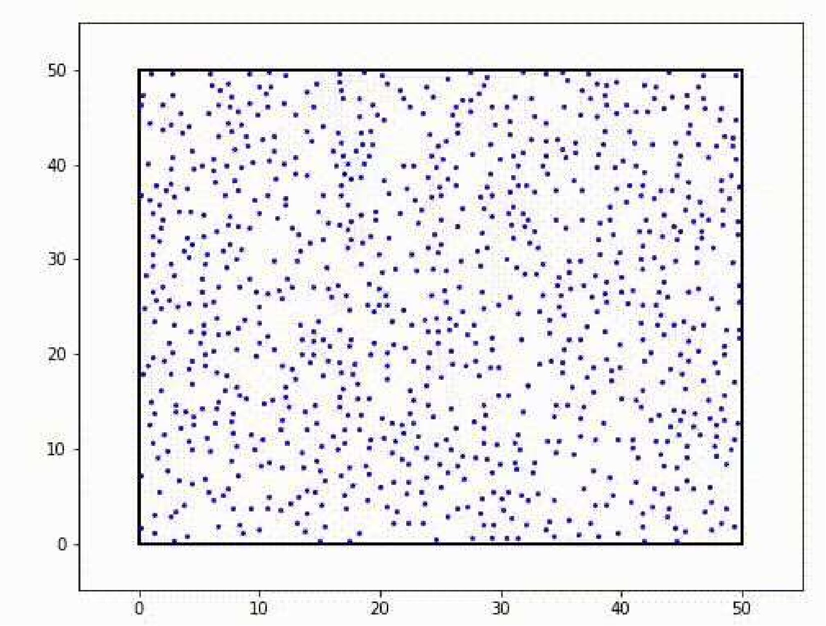
What is temperature?

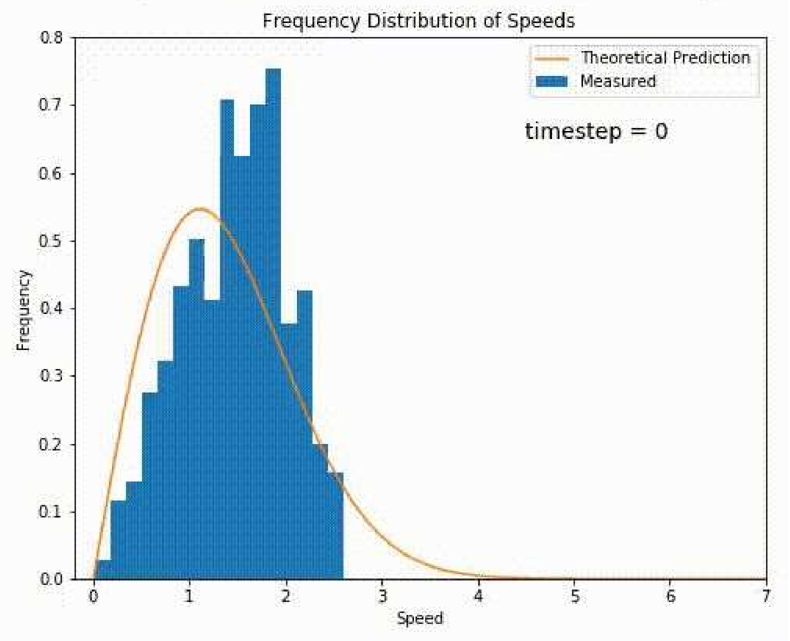
(1)

(2)

Contextualize | Heat Diffusion Problem (2)

What is temperature? - Statistical Mechanics view





(Left) Random atom movement. (Right) Distribution of Speed of Atom overlaid by Maxwell-Boltzman distribution function

Contextualize | Heat Diffusion Problem (3)

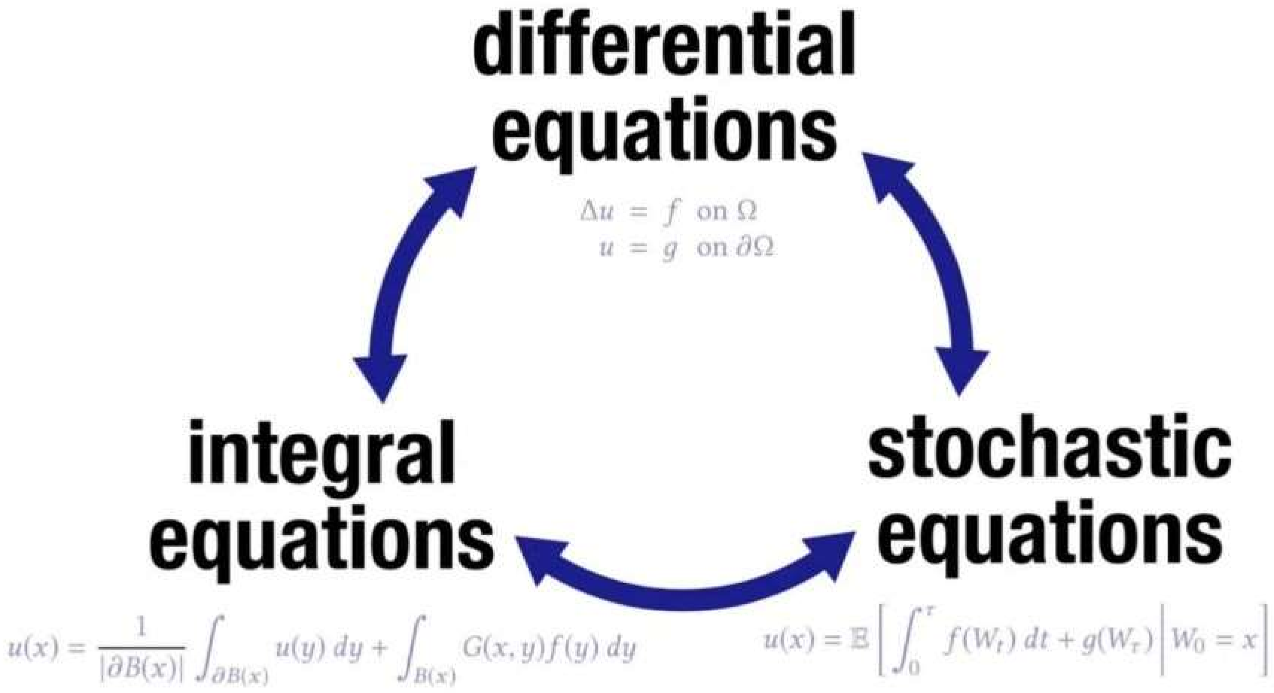
How do we describe heat spreading?

(1) Scalar of field that describe how heat flows

$$\nabla^2 T = \frac{d^2}{dx^2}T + \frac{d^2}{dy^2}T + \frac{d^2}{dz^2}T = f$$

(2) Atoms of different kinetic energy level spreading and mixing throughout the substances

Contextualize | Heat Diffusion Problem (4)



Boundary Condition

Dirichlet BC

$$U_{\Gamma} = f(x)$$

- Specifies value at boundary
- Ex: Temperature at boundary
- *Walk on Star* works only with this BC.

Neumann BC

$$\delta / \delta x \, U_{\Gamma} = f(x)$$

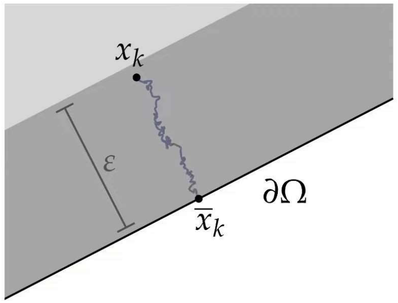
- Specifies derivative of value at boundary
- Ex: Heat flux / Insulator at boundary

Boundary Condition | WoS and Neumann BC

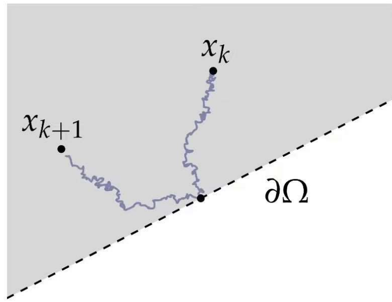
- We can view Neumann boundary as *reflector*

- *Naive WoS is Slow with Neumann BC*. Particle can bounce after a while before it actually reach a Dirichlet boundary.

  Can we increase each step size?



Dirichlet



Neumann

Walk-on-Stars | Going bigger than Sphere

This paper derives the equation for estimator in a star-shaped region that is bigger than a sphere.

Walk on Stars Estimator

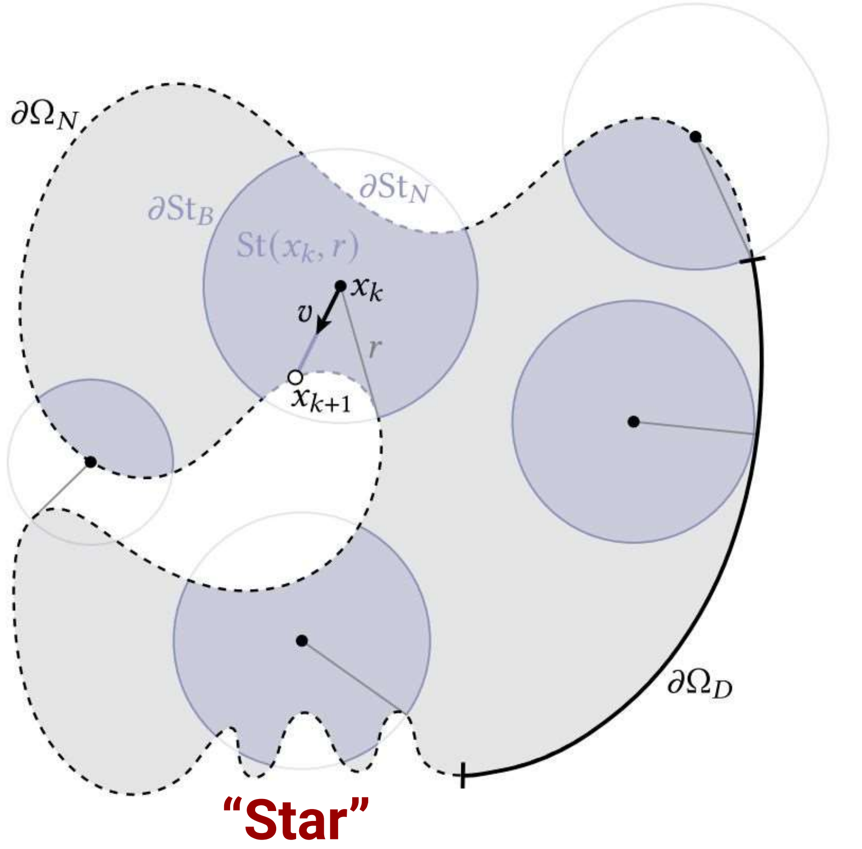
A recursive single-sample estimator for Equation 17 is given by

$$\widehat{u}(x_k) := \underbrace{\frac{P^B(x_k, x_{k+1})\,\widehat{u}(x_{k+1})}{\alpha(x_k)\,p^{\partial \mathrm{St}(x_k,r)}(x_{k+1})} - \frac{G^B(x_k, z_{k+1})\,h(z_{k+1})}{\alpha(x_k)\,p^{\partial \mathrm{St}_N(x_k,r)}(z_{k+1})}}_{\text{boundary}}$$

$$+ \underbrace{\frac{G^B(x_k, y_{k+1})\,f(y_{k+1})}{\alpha(x_k)\,p^{\mathrm{St}(x_k,r)}(y_{k+1})}}_{\text{interior}}, \qquad (18)$$

where

- the points $x_{k+1} \in \partial\mathrm{St}$, $z_{k+1} \in \partial\mathrm{St}_N$, and $y_{k+1} \in \mathrm{St}$ are sampled from the probability densities $p^{\partial\mathrm{St}}$ (Section 4.4), $p^{\partial\mathrm{St}_N}$ (Section 4.5), and p^{St} (Section 4.6), *resp.*
- r is chosen so that $\mathrm{St}(x_k, r)$ is star-shaped (Section 4.4).



Walk-on-Stars | Why it works

1. We want this integral over region A inside our boundary condition Γ

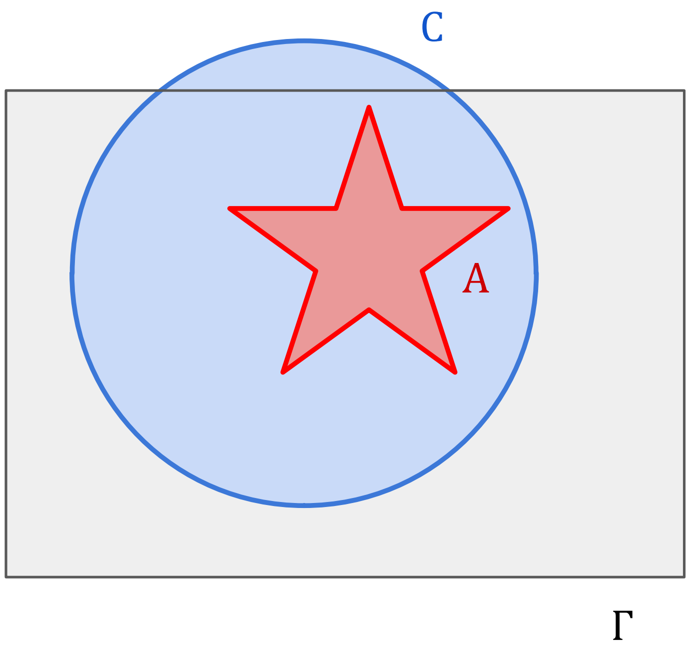
$$u(x) = \int_{\delta A} P^{A}(x, y)\, u(y)\, dy\,.$$

2. But if region A is not a sphere or rectangle, Poisson kernel P^{A} is hard to analyze.

3. However, if A is inside a region C, we can use its Poisson P^{C} instead.

$$u(x) = \int_{\delta A} P^{C}(x, y)\, u(y)\, dy\,.$$

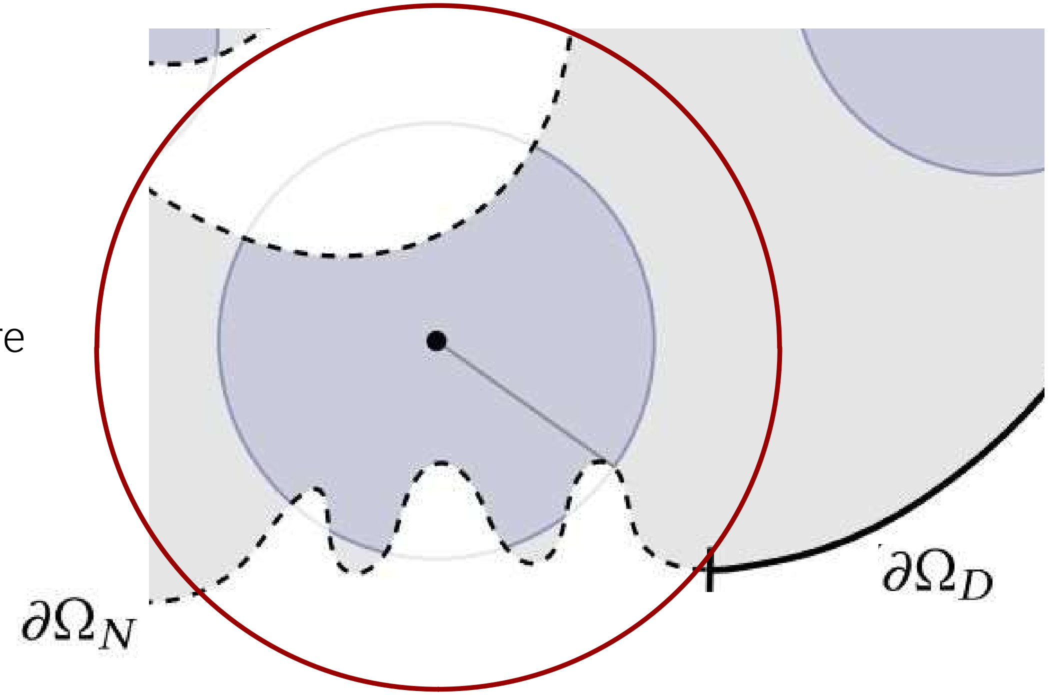
4. And we can just choose C to be a simple shape like a sphere.



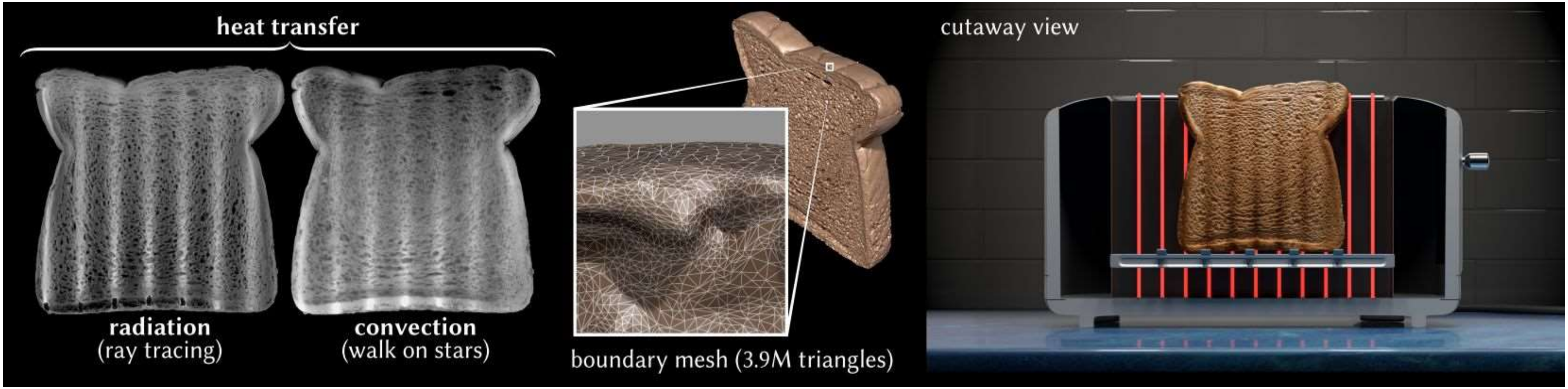
Walk-on-Stars | Star-shaped subdomain (This will be quizzed.)

WoSt uses 2 regions

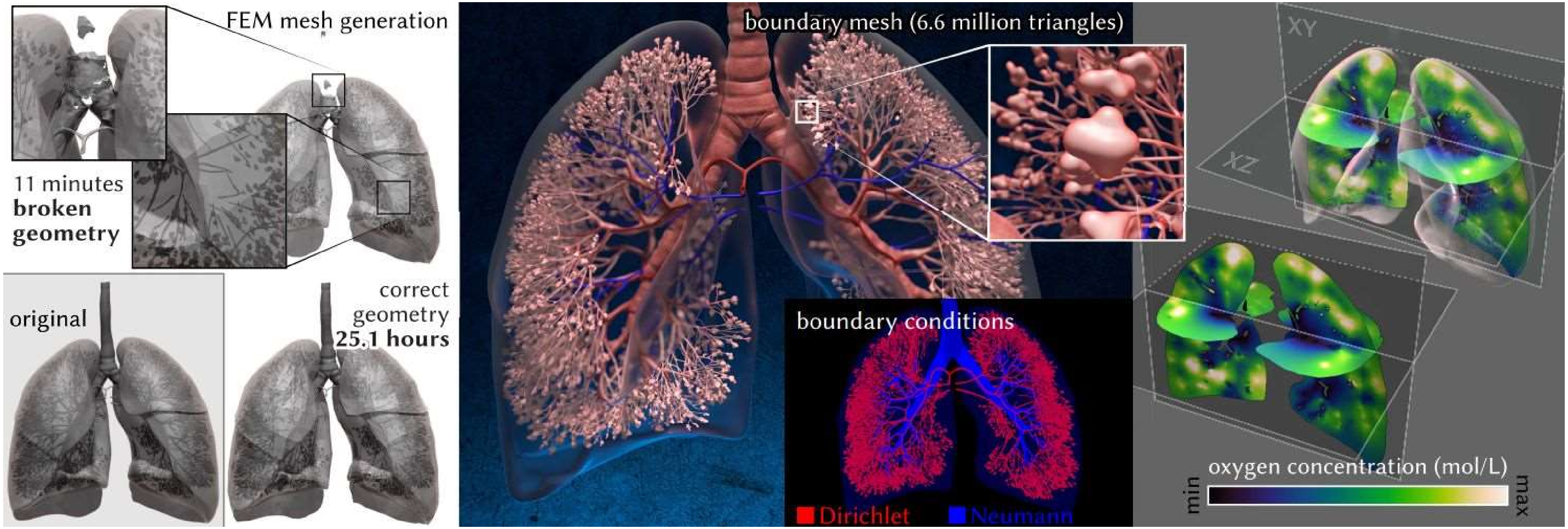
- *Sphere region* which touch the closest Dirichlet boundary
- *"Star" region* which is the biggest sphere whose ray cast only intersect the boundary once.
    - We take a new sample from this region.



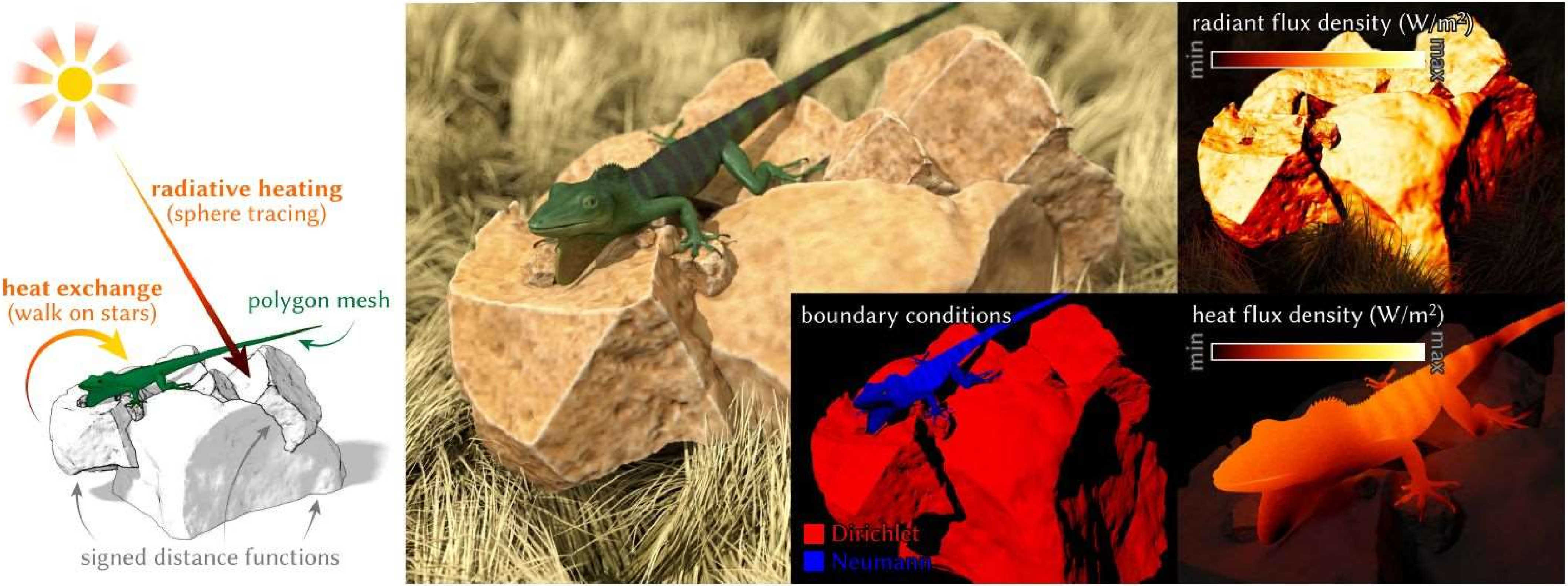
Application | Toasting bread faster



Application | Comparison with FEM



Application | Combining with Ray Tracing



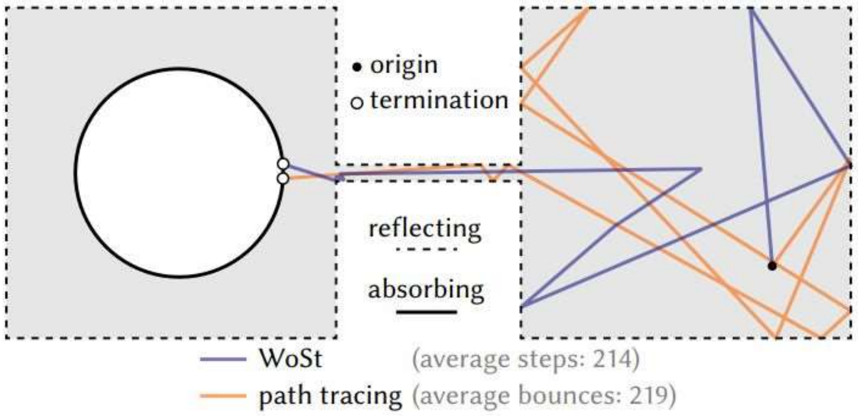
Limitations

1. Assume perfect Dirichlet and Neumann BC

   Robin BC $a\, U_\Gamma + b\, \delta/\delta x\, U_\Gamma = f(x)$

   In a realistic scene, no material is truly a perfect absorber or reflector.

2. Limited to only 1 type of PDE — Poisson equation



3. Keyhole problem